Lie algebras and polynomials in one variable

1992 J. Phys. A: Math. Gen. 25 L1087

(http://iopscience.iop.org/0305-4470/25/18/001)

J. Phys. A: Math. Gen. 25 (1992) L1087–L1093. Printed in the UK

LETTER TO THE EDITOR

Lie algebras and polynomials in one variable

A Turbiner†‡

CPT, CNRS-Luminy, Marseille, F-13288, France, and (present address) Theoretische Physik, ETH-Honggerberg, CH-8093 Zurich, Switzerland

Received 4 June 1992

Abstract. A classification of linear differential and difference equations in one variable having polynomial solutions (the generalized Bochner problem) is presented. The idea of the approach is based on consideration of the eigenvalue problem for a polynomial in generators of the algebra $sl_2(\mathbb{R})$ (for differential equations) or $sl_2(\mathbb{R})_q$ (for finite-difference equations) given in the finite-dimensional 'projectivized' representation. Connection with the recently discovered quasi-exactly-solvable Schrödinger equations is discussed.

In 1929 S Bochner asked about the classification of differential equations

$$T\varphi(x) = \epsilon\varphi(x) \tag{1}$$

where T is a linear differential operator of kth order in one real variable $x \in \mathbb{R}$ and ϵ is the spectral parameter, having a sequence of *orthogonal* polynomial solutions (see Littlejohn (1988)).

Definition. Let us give the name of *generalized Bochner problem* to the problem of classifying the differential equations (1) having $(n+1)$ eigenfunctions in the form of a polynomial of order not higher than n.

In Turbiner (1992) a general method has been formulated for generating linear differential operators, linear matrix differential operators and linear finite-difference operators in one and several variables, for which corresponding eigenvalue problem possesses polynomial solutions. This method is closely connected to the finite-dimensional, 'projectivized' representations of Lie algebras (Turbiner 1992). In this letter it will be shown that this method provides both necessary and sufficient conditions for the description of general linear finite-order differential and finite-difference equations in one variable possessing polynomial solutions.

Consider the space of all polynomials of order n

$$\mathcal{P}_n = \langle 1, x, x^2, \ldots, x^n \rangle \tag{2}$$

where n is a non-negative integer and $x \in \mathbb{R}$.

† On leave of absence from: Institute for Theoretical and Experimental Physics, Moscow 117259, Russia.
‡ E-mail: TURBINER@CERNVM or TURBINER@VXCERN.CERN.CH

0305-4470/92/181087+07$04.50

Definition. Let us call a linear differential operator of the kth order, T_k *quasi-exactly-solvable*, if it preserves the space $\mathcal{P}_n$. Correspondingly, the operator E_k, which preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots$ of spaces of all polynomials, is called *exactly-solvable.*

Lemma 1. (i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator of kth order T_k, can be represented by a kth degree polynomial of the operators

$$J^+ = x^2\partial_x - nx \qquad J^0 = x\partial_x - \tfrac{1}{2}n \qquad J^- = \partial_x \tag{3}$$

(the operators (3) obey the $\mathrm{sl}_2(\mathbb{R})$ commutation relations †). If $n \leqslant (k-1)$, the part of the quasi-exactly-solvable operator T_k containing derivatives up to order n can be represented by an nth degree polynomial in the generators (3). (ii) Conversely, any polynomial in (3) is quasi-exactly solvable. (iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $E_k \subset T_k$.

Comment 1. If we define the universal enveloping algebra U_g of a Lie algebra g as the algebra of all polynomials in generators, then T_k at $k < n+1$ is simply an element of the universal enveloping algebra $U_{\mathrm{sl}_2(\mathbb{R})}$ of the algebra $\mathrm{sl}_2(\mathbb{R})$ taken in representation (3). If $k \geqslant n+1$, then T_k is represented as an element of $U_{\mathrm{sl}_2(\mathbb{R})}$ plus $B\mathrm{d}^{n+1}/\mathrm{d}x^{n+1}$, where B is any linear differential operator of the order not higher than $(k-n-1)$.

Since $\mathrm{sl}_2(\mathbb{R})$ is a graded algebra, let us introduce the grading of generators (3)

$$\deg(J^+) = +1 \qquad \deg(J^0) = 0 \qquad \deg(J^-) = -1 \tag{4}$$

and hence

$$\deg\left[(J^+)^{n_+}(J^0)^{n_0}(J^-)^{n_-}\right] = n_+ - n_-. \tag{5}$$

The grading allows to classify the operators T_k in a Lie-algebraic sense.

Lemma 2. A quasi-exactly-solvable operator $T_k \subset U_{\mathrm{sl}_2(\mathbb{R})}$ has no terms of positive grading, iff it is an exactly-solvable operator.

Theorem 1. Let n be a non-negative integer. Take the eigenvalue problem for a linear differential operator of the kth order in one variable

$$T_k\varphi = \varepsilon\varphi \tag{6}$$

where T_k is symmetric. The problem (6) has $(n+1)$ linear independent eigenfunctions in the form of polynomials in the variable x of order not higher than n, iff T_k is quasi-exactly-solvable. The problem (6) has an infinite sequence of polynomial eigenfunctions, iff the operator is exactly-solvable.

Comment 2. The 'if' part of the first statement is obvious. The 'only if' part is a direct corollary of lemma 1.

† The representation (3) is one of the above-mentioned 'projectivized' representations (see Turbiner (1992)).

This theorem gives a general classification of differential equations

$$\sum_{j=0}^{k} a_j(x)\varphi^{(j)}(x) = \varepsilon\varphi(x) \tag{7}$$

having at least one polynomial solution in x, thus resolving the generalized Bochner problem. The coefficient functions $a_j(x)$ should have the form

$$a_j(x) = \sum_{i=0}^{k+j} a_{j,i} x^i \,. \tag{8}$$

The coefficients $a_{j,i}$ can be expressed through the coefficients of the kth degree polynomial element of the universal enveloping algebra $U_{\mathrm{sl}_2(\mathbb{R})}$. Hence the number of free parameters of the polynomial solutions is defined by the number of parameters of a general kth degree polynomial element of $U_{\mathrm{sl}_2(\mathbb{R})}$ and this number is equal to†

$$\mathrm{par}(T_k) = (k+1)^2 \tag{9}$$

(we denote the number of free parameters of the operator T_k by the symbol $\mathrm{par}(T_k)$). For the case of an infinite sequence of polynomial solutions the expression (8) simplifies to

$$a_j(x) = \sum_{i=0}^{j} a_{j,i} x^i \tag{10}$$

in agreement with known results (see e.g. Littlejohn (1988)). In this case

$$\mathrm{par}(E_k) = \frac{(k+1)(k+2)}{2}. \tag{11}$$

One may ask a more general question: which non-degenerate linear differential operators have finite-dimensional invariant sub-space of the form

$$\langle \alpha(x), \alpha(x)\beta(x), \ldots, \alpha(x)\beta(x)^n \rangle \tag{12}$$

where $\alpha(x)$ is a function and $\beta(x)$ is a diffeomorphism of the line? Such operators are obtained from those of theorem 1 by the change of variable $x \mapsto \beta(x)$ and the 'gauge' transformation $\varphi(x) \mapsto \alpha(x)\varphi(x)$.

Let us consider the case of second order differential equations (7) possessing polynomial solutions. As follows from theorem 1 the corresponding differential operator should be quasi-exactly-solvable and can be represented as

$$\begin{aligned} T_2 = {} & c_{++}J^+J^+ + c_{+0}J^+J^0 + c_{+-}J^+J^- + c_{0-}J^0J^- + c_{--}J^-J^- + c_+J^+ \\ & + c_0J^0 + c_-J^- + c \end{aligned} \tag{13}$$

where $c_{\alpha\beta}$, c_α, $c \in \mathbb{R}$, while $\mathrm{par}(T_2) = 9$. If $c_{++} = c_{+0} = c_+ = 0$, then T_2 becomes exactly-solvable E_2 (lemma 2) and $\mathrm{par}(E_2) = 6$.

† Counting free parameters, one should introduce a certain ordering of the generators to avoid double counting because of commutation relations. Also the quadratic Casimir operator and the double-sided ideal generated by it should not be taken into account.

Lemma 3. If the operator (13) is such that

$$c_{++} = 0 \qquad c_+ = (\tfrac{1}{2}n - m)c_{+0} \qquad \text{for some } m = 0, 1, 2, \ldots \tag{14}$$

then the operator T_2 preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$, $\mathrm{par}(T_2) = 7$.

In fact, lemma 3 means that $T_2(J^\alpha(n), c_{\alpha\beta}, c_\alpha)$ can be rewritten as $T_2(J^\alpha(m), c'_{\alpha\beta}, c'_\alpha)$. As a consequence of lemma 3 and theorem 1 there are polynomials of order n and order m among polynomial solutions of (7).

Remark. From the Lie-algebraic point of view lemma 3 implies the existance of representations of second-degree polynomials in the generators (3) possessing two invariant sub-spaces. In general, if n is not a non-negative integer in (3) (and hence (3) becomes infinite-dimensional), then among representations of kth degree polynomials in the generators (3), lying in the universal enveloping algebra, there are representations possessing $0, 1, 2, \ldots, k-1$ invariant sub-spaces.

Substituting (3) into (13) and then into (6), we obtain

$$P_4(x)\partial_x^2\varphi(x) + P_3(x)\partial_x\varphi(x) + P_2(x)\varphi(x) = \varepsilon\varphi(x) \tag{15}$$

where the $P_j(x)$ are polynomials of jth order with coefficients related to $c_{\alpha\beta}$, c_α and n (see (10)). In general, problem (15) has $(n+1)$ polynomial solutions. According to lemma 1, if $n = 1$, then a more general spectral problem than (15), possessing two polynomial solutions of the form $(ax + b)$, arises

$$F_3(x)\partial_x^2\varphi(x) + Q_2(x)\partial_x\varphi(x) + Q_1(x)\varphi(x) = \varepsilon\varphi(x) \tag{16}$$

where F_3 is an arbitrary real function and $Q_j(x), j = 1, 2$ are polynomials of order j. If $n = 0$ (one polynomial solution), the spectral problem (15) becomes

$$F_2(x)\partial_x^2\varphi(x) + F_1(x)\partial_x\varphi(x) + Q_0\varphi(x) = \varepsilon\varphi(x) \tag{17}$$

where $F_{2,1}(x)$ are arbitrary real functions and Q_0 is a real constant. After the transformation

$$t : \varphi \mapsto \varphi(x(z))\mathrm{e}^{A(z)} \tag{18}$$

where $z \mapsto x(z)$ is a diffeomorphism of the line and $A(z)$ is a certain real function, one can reduce (15)–(17) to the Sturm–Liouville problem

$$(-\partial_z^2 + V(z))\varphi = \varepsilon\varphi \tag{19}$$

with the potential

$$V(z) = (A')^2 + A'' + P_2(x(z))$$

where $A = \int (P_3/P_4)\,\mathrm{d}x - \log z'$ for (15). If the functions (18), obtained after transformation, belong to L_2-space, we reproduce recently discovered quasi-exactly-solvable problems (Turbiner 1988a, b), for which a finite number of eigenstates are found algebraically. For example

$$T_2 = -4J^0J^- + 4aJ^+ + 4bJ^0 - 2(n+1)J^- \tag{20}$$

leads to the spectral problem (19) with the potential

$$V(z) = a^2 z^6 + 2abz^4 + (b^2 - (4n+3)a)z^2 \tag{21}$$

for which the first $(n+1)$ eigenfunctions, even in x, can be found algebraically.

It is worth noting that the use of (16) as input leads to the one-functional family of Schrödinger operators with two explicitly known eigenstates. One such operator has been described by Jatkar *et al* (1989), where the authors confusingly stated the non-existance of an underlying $\mathrm{sl}_2(\mathbb{R})$ algebra.

Taking different exactly-solvable operators E_2 for the eigenvalue problem (7) one can reproduce the equations having the Hermite, Laguerre, Legendre and Jacobi polynomials as solutions (Turbiner 1992)†. Also under special choices of the general element E_4, one can reproduce all known fourth order differential equations giving rise to infinite sequences of orthogonal polynomials (see Littlejohn (1988) and other papers in that volume).

Recently, Gonzalez-Lopez *et al* (1992) gave the complete description of second-order polynomial elements of $U_{\mathrm{sl}_2(\mathbb{R})}$ leading to the square-integrable eigenfunctions of the Sturm–Liouville problem (19) after transformation (18). Consequently, for second-order ordinary differential equations (15) the combination of this result and theorem 1 gives a general solution of the Bochner problem as well as the more general problem of the classification of equations possessing a finite number of *orthogonal* polynomial solutions.

Now let us proceed to finite-difference equations in one variable. The generalized Bochner problem is defined in the same way as for differential equations. The only difference is the operator T in the problem (1) is a linear finite-difference operator (see definition below). For the case of one real variable, a solution of the classification problem is very similar to the case of ordinary differential equations.

Let us introduce the finite-difference analogue of the generators (3) (Ogievetski and Turbiner 1991)

$$\tilde{J}^+ = x^2 D - \{n\}x \qquad \tilde{J}^0 = xD - \hat{n} \qquad \tilde{J}^- = D \tag{22}$$

where $\hat{n} \equiv \{n\}\{n+1\}/\{2n+2\}$, $\{n\} = (1-q^n)/(1-q)$ is the quantum symbol, q is a number characterizing the deformation, $Dz = 1 + qzD$ and $Df(z) = (f(z) - f(qz))/(1-q)z + f(qz)D$ is a shift or finite-difference operator (or the so called Jackson symbol (Exton 1983)). The operators (22) are obeyed by q-deformed commutation relations corresponding to the quantum $\mathrm{sl}_2(\mathbb{R})_q$ algebra (this is the so called 'Witten's second deformation' of sl_2 in the classification of C Zachos (Zachos 1991)). If $q \to 1$, the q-commutation relations reduce to the standard $\mathrm{sl}_2(\mathbb{R})$ ones. If n is a non-negative integer, the representation (22) becomes finite-dimensional!

Analogously, as for differential operators, one can introduce quasi-exactly-solvable and exactly-solvable operators.

Definition. Let us call a linear difference operator of the kth order, $\tilde{T}_k$ *quasi-exactly-solvable*, if it preserves the space $\mathcal{P}_n$. The operator $\tilde{E}_k$, which preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots$ of spaces of all polynomials, is called *exactly-solvable.*

† For instance, if $a = 0$ in (20), the equation (15) becomes the Hermite equation (after some substitution).

The analogue of the lemma 1 holds.

Lemma 4. (i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator of kth order $\tilde{T}_k$, can be represented by a kth degree polynomial of the operators (22). If $n \leqslant (k-1)$, the part of the quasi-exactly-solvable operator $\tilde{T}_k$ containing derivatives up to order n can be represented by a nth degree polynomial in the generators (22). (ii) Conversely, any polynomial in (22) is quasi-exactly solvable. (iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $\tilde{E}_k \subset \tilde{T}_k$.

Similarly to $\mathrm{sl}_2(\mathbb{R})$, one can introduce the grading of the generators (22) of $\mathrm{sl}_2(\mathbb{R})_q$ (see (4)) and, hence, of monomials of the universal enveloping $U_{\mathrm{sl}_2(\mathbb{R})_q}$ (see (5)). The analogue of lemma 2 holds.

Lemma 5. A quasi-exactly-solvable operator $\tilde{T}_k$ has no terms of positive grading, iff it is an exactly-solvable operator.

Theorem 2. Take the eigenvalue problem for a linear difference operator of the kth order in one variable

$$\tilde{T}^k \varphi(x) = \varepsilon\varphi(x) \tag{23}$$

where $\tilde{T}_k$ is symmetric. The problem (23) has $(n+1)$ linear independent eigenfunctions in the form of polynomials in the variable x of order not higher than n, iff $\tilde{T}_k$ is quasi-exactly-solvable. The problem (23) has an infinite sequence of polynomial eigenfunctions, iff the operator is exactly-solvable $\tilde{E}_k$.

This theorem gives a general classification of finite-difference equations

$$\sum_{j=0}^{k} \tilde{a}_j(x) D^j \varphi(x) = \varepsilon\varphi(x) \tag{24}$$

having polynomial solutions in x. The coefficient functions should have the form

$$\tilde{a}_j(x) = \sum_{i=0}^{k+j} \tilde{a}_{j,i} x^i. \tag{25}$$

The coefficients $\tilde{a}_{j,i}$ are related to the coefficients of the kth degree polynomial element of the universal enveloping algebra $U_{\mathrm{sl}_2(\mathbb{R})_q}$ and the number of free parameters in the polynomial solutions is defined by the number of free parameters in a general kth order polynomial in the generators (22)†. This number is given by

$$\mathrm{par}(\tilde{T}_k) = (k+1)^2 + 1$$

† For the quantum $\mathrm{sl}_2(\mathbb{R})_q$ algebra there are no polynomial Casimir operators (see e.g. Zachos 1991). However, in the representation (22) the relationship between generators analogous to the quadratic Casimir operator

$$q\tilde{J}^+\tilde{J}^- - \tilde{J}^0\tilde{J}^0 + (\{n+1\} - \hat{n})\tilde{J}^0 = \hat{n}(\hat{n} - \{n+1\})$$

appears. This expression becomes the standard Casimir operator in the limit $q \to 1$.

(particularly, $\mathrm{par}(\tilde{T}^2) = 10$). For the case of an infinite sequence of polynomial solutions, the formula (25) simplifies to

$$\tilde{a}_j(x) = \sum_{i=0}^{j} \tilde{a}_{j,i} x^i \tag{26}$$

and the number of free parameters is given by

$$\mathrm{par}(\tilde{E}_k) = \frac{(k+1)(k+2)}{2} + 1$$

(particularly, $\mathrm{par}(\tilde{E}^2) = 7$). The increase in the number of free parameters compared to ordinary differential equations is due to the presence of the deformation parameter q.

The analogue of lemma 3 also holds.

Lemma 6. If the operator $\tilde{T}_2$ (see (13)) is such that

$$\tilde{c}_{++} = 0 \qquad \tilde{c}_{+} = (\hat{n} - \{m\})\tilde{c}_{+0} \qquad \text{for} \quad m = 0, 1, 2, \ldots \tag{27}$$

then the operator $\tilde{T}_2$ preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$ and polynomial solutions in x with 8 free parameters occur.

In Turbiner (1992) one can find the description of the equations having as the eigenfunctions the q-deformed Hermite, Laguerre, Legendre and Jacobi polynomials in this approach.

Since in the limit q tends to one, lemmas 4, 5, 6 and theorem 2 coincide with lemmas 1, 2, 3 and theorem 1, respectively. Thus the case of differential equations in one variable can be considered as a particular case of finite-difference ones.

In closing, I would like to thank to L Michel, V Ovsienko, S Tabachnikov and, especially, V Arnold and M Shubin for numerous useful discussions. Also I am very grateful for hospitality extended to me by the Theoretische Physik, ETH where this work has been completed.

References

Exton H 1983 *q-Hypergeometrical Functions and Applications* (Chichester: Ellis Horwood)
Gonzales-Lopez A, Kamran N and Olver P 1992 *Preprint* School of Mathematics, University of Minnesota (Invited talk at the session of AMS no 873, March 20–21, Springfield, USA)
Jatkar D P, Nagaraja Kumar C and Khare A 1989 *Phys. Lett.* **142A** 200
Littlejohn L L 1988 *Lecture Notes in Mathematics 1329* ed M Alfaro *et al* (Berlin: Springer) p 98
Ogievetski O and Turbiner A 1991 *Preprint* CERN-TH: 6212/91
Turbiner A V 1988a *Sov. Math. Funk. Analysis i ego Prilogenia* **22** 92
—— 1988b *Commun. Math. Phys.* **118** 467
—— 1992 *J. Math. Phys.* in press (*Preprint* CPT-91/P.2628)
Zachos C 1991 *Proc. Conf. Deformation Theory of Algebras and Quantization with Applications to Physics, Contemporary Mathematics* ed J Stasheff and M Gerstenhaber